

# Fingerprint Image Compression Standard Based on Wave Atoms Decomposition and Self Organizing Feature Map

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**Abstract**— Multiresolution analysis tools have been successfully applied to fingerprint image compression for more than a decade. Fingerprint image compression and reconstruction standards used by the US Federal Bureau of Investigation (FBI) are based upon the popular 9/7 discrete wavelet transform (DWT). A novel fingerprint image compression technique based on recently proposed multiresolution analysis tool i.e. wave atoms decomposition is proposed. Wave atoms decomposition has been specifically designed for enhanced representation of oscillatory patterns and to convey temporal and spatial information more precisely. Our proposed compression scheme is based upon vector quantization of decomposed wave atoms representation of fingerprint images using self organizing feature map (SOM). The quantized information is later encoded using arithmetic entropy scheme. The proposed compression scheme outperforms the FBI fingerprint image compression standard, the wavelet scalar quantization (WSQ). Data mining, law enforcement, border security, and forensic applications can potentially benefit from our proposed compression technique.

## INTRODUCTION

The fundamental goal of image compression is to obtain the best possible image quality at an allocated storage capacity. Law enforcement, border security and forensic applications are some crucial fields where fingerprint image compression plays an important role. Emergence of protocols and commercially available products have prompted police forces to use Automated Fingerprint Identification Systems (AFIS) during criminal investigations. The FBI deals with a massive collection of fingerprint database that contains more than 200 million cards and is growing at the rate of 30,000-50,000 new cards daily [1]. The archive consists of inked impressions on paper cards. A single card contains 14 different images: 10 rolled impression of each finger, duplicate (flat) impression of thumb and simultaneous impression of all fingers together.

Fingerprint images are digitized at a resolution of 500 pixels per inch with 256 gray levels. Therefore a single fingerprint card requires approximately 10 MB of storage.

The gray level images tender a refined natural appearance to humans than black and white images and allow a higher level of subjective discrimination for fingerprint examiners. The investigation of an efficient compression standard, that can significantly reduce the image size while retaining its minutiae (ridges, endings and bifurcations) information, is justified in conjunction with the size of FBI fingerprint database. In addition to the considerable saving in storage capacity, fingerprint image compression is also desired for effortless archiving and sweeping reduction in transmission bandwidth.

FBI compression standard has incorporated biorthogonal 9/7 DWT filter pair for highly reliable fingerprint compression and reconstruction since 1993. DWT is a widely used multiresolution analysis tool due to its capability of space-frequency decomposition of images [2], energy compaction of low frequency sub-bands, space localization of high frequency sub-bands, flexibility in time frequency tiling, attractive properties for extracting features from nonstationary signals, ability to match human vision spectral properties, possibility of using short filters and the absence of blocking artifacts. Wavelet packets facilitate a flexible representation by allowing decompositions at every node of the tree resulting in an explicit structure for specific applications. Image analysis using discrete wavelet transform is performed using a pair of quadrature mirror filter (QMF) and a dual quadrature mirror filter (DQMF). QMF and DQMF are further decomposed into four sets of floating point coefficients:  $h0(Lo\_D)$ ,  $g0(Hi\_D)$ ,  $h1(Lo\_R)$  and  $g1(Hi\_D)$  that define the wavelet and scaling functions for forward DWT and inverse DWT respectively. Table I details the filter coefficients used for image decomposition and reconstruction.

Fingerprint images are decomposed using a 2D DWT which is applied by means of separability approach along its rows and columns alternatively resulting into four smaller subsets. Selected subsets are iteratively decomposed until desired sub-bands are obtained. These subbands are quantized and coded using different coding techniques. Researchers have

TABLE I. FILTER COEFFICIENTS FOR FBI COMPRESSION STANDARD

Filter Type	Filter Coefficients
h0 (Lo_Dec)	[0.03783 -0.02385 -0.11062 0.37740 0.85270 0.37740 -0.11062 -0.02385 0.03783]
g0 (Hi_D)	[0.06454 -0.04069 -0.41809 0.78849 -0.41809 -0.04069 0.06454]
h1 (Lo_D)	[-0.06454 -0.04069 0.41809 0.78849 0.41809 -0.04069 -0.06454]
g1 (Hi_D)	[0.03783 0.02385 -0.11062 -0.37740 0.85270 -0.37740 -0.11062 0.02385 0.03783]

proposed various techniques for iterative decomposition such as FBI’s 64- subband [3] and Kasaei *et al*’s 73-Subband decomposition [4]. For improved sub-band decomposition entropy based best basis selection (EBBS) algorithm [5] has been proposed. Recently proposed fingerprint image compression schemes use genetic algorithm [6-7] to evolve wavelet and scaling numbers for each level of decomposition. In this paper we propose a generic algorithm for fingerprint image compression using wave atoms decomposition. Extensive experiments are performed and our proposed method significantly outperforms the traditional WSQ FBI compression standard.

The remainder of this paper is divided into 6 sections. Section 2 discusses the wave atoms transform along with its implementation details and section 3 briefly explains the self organizing feature map. The proposed method for fingerprint image compression is described in section 4. Experimental results are discussed in section 5 followed by conclusion, acknowledgment and references.

WAVE ATOMS DECOMPOSTION

Fourier series decomposes a periodic function into a sum of simple oscillating functions, namely *sines* and *cosines*. In a Fourier series sparsity is destroyed due to discontinuities (Gibbs Phenomenon) and it requires a large number of terms to reconstruct a discontinuity precisely. Development of new mathematical and computational tools based on multiresolution analysis is a novel concept to overcome limitations of Fourier series. Many fields of contemporary science and technology benefit from multiscale, multiresolution analysis tools for maximum throughput, efficient resource utilization and accurate computations. Multiresolution tools render robust behavior to study information content of images and signals in the presence of noise and uncertainty.

Wavelet transform is a well known multiresolution analysis tool capable of conveying accurate temporal and spatial information. Wavelet transform has been profusely used to address problems in; data compression, pattern recognition, image reconstruction and computer vision. Wavelets better represent objects with point singularities in 1D and 2D space but fail to deal with singularities along curves in 2D. Discontinuities in 2D are spatially distributed which leads to extensive interaction between discontinuities and many terms of wavelet expansion. Therefore wavelet representation does not offer sufficient sparseness for image

analysis. Following the introduction of wavelet transform, research community has witnessed intense efforts for development of ridgelets [8], contourlets [9], and curvelets [10]. These tools have better directional and decomposition capabilities than wavelets.

Wave atoms are a recent addition to the collection of mathematical transforms for harmonic computational analysis. Wave atoms are a variant of 2D wavelet packets that retain an isotropic aspect ratio. Wave atoms have a sharp frequency localization that cannot be achieved using a filter bank based on wavelet packets and offer a significantly sparser expansion for oscillatory functions than wavelets, curvelets and Gabor atoms. Wave atoms capture the coherence of patterns across and along oscillations whereas curvelets capture coherence along oscillations only. Wave atoms precisely interpolate between Gabor atoms [14] (constant support) and directional wavelets [15] (wavelength ~ diameter) in the sense that the period of oscillations of each wave packet (wavelength) is related to the size of essential support by parabolic scaling i.e. wavelength ~ (diameter)<sup>2</sup>.

Two distinct parameters  $\alpha, \beta$  represent decomposition and directional ability and are sufficient for indexing all known forms of wave packet architectures namely wavelets, Gabor, ridgelets, curvelets and wave atoms. Wave atoms are defined for  $\alpha=\beta=1/2$  and essential support of wave packet in space (left) and in frequency (right) is shown in Fig. 1.  $\alpha$  indexes the multiscale nature of the transform, from  $\alpha = \mathbf{0}$  (uniform) to  $\alpha = \mathbf{1}$  (dyadic).  $\beta$  measures the wave packet’s directional selectivity ( $\mathbf{0}$  and  $\mathbf{1}$  indicate best and poor selectivity respectively). Wave atoms represent a class of wavelet packets where directionality is sacrificed at the expense of preserving sparsity of oscillatory patterns under smooth diffeomorphisms.

A. 1D Discrete Wave Atoms

Wave atoms are constructed from tensor products of adequately chosen 1D wave packets. Let  $\psi_{m,n}^j(x)$  represent a one-dimensional family of wave packets, where  $j, m \geq 0$ , and  $n \in \mathbf{Z}$ , centered in frequency around  $\pm \omega_{j,m} = \pm \pi 2^j m$ , with  $C_1 2^j \leq m \leq C_2 2^j$  and centered in space around  $x_{j,n} = 2^j n$ . One-dimensional version of the parabolic scaling states that the support of each bump of  $\psi_{m,n}^j(\omega)$  is of length  $O(2^j)$  while  $\omega_{j,m} = O(2^{2j})$ .

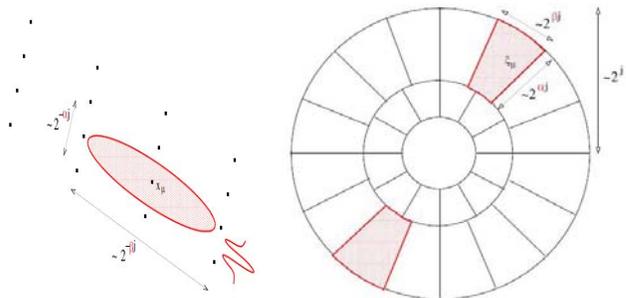


Figure 1. Wave Atoms Tiling in Space and in Frequency[12].

Dyadic dilates and translates of  $\hat{\psi}_m^0$  on the frequency axes are combined and basis function, written as:

$$\psi_{m,n}^j(x) = \psi_m^j(x - 2^{-j}n) = 2^{j/2} \psi_m^0(2^j x - n). \quad (1)$$

The coefficients  $c_{j,m,n}$ , for each wave number  $\omega_{j,m,n}$ , are obtained as a decimated convolution at scale  $2^j$ .

$$c_{j,m,n} = \int \psi_m^j(x - 2^{-j}n) u(x) dx. \quad (2)$$

By Plancherel's theorem,

$$c_{j,m,n} = \int e^{i2^{-j}nw} \overline{\hat{\psi}_m^j(\omega)} \hat{u}(\omega) d\omega. \quad (3)$$

If the function  $u$  is discretized at  $x_k = kh$ ,  $h=1/N$ ,  $k=1\dots N$ , then with a small truncation error (3) is modified as:

$$c_{j,m,n}^D = \sum_{k=2\pi(-N/2+1/N/2)} e^{i2^{-j}nk} \overline{\hat{\psi}_m^j(k)} \hat{u}(k). \quad (4)$$

Since the data is supported inside two disjoint intervals of size  $2^{j+1}\pi$ , symmetric about origin ( $2^{j+1}$  points) instead of an interval of length  $2^j \times 2\pi$ , sum(4) is computed using a reduced inverse FFT inside an interval of size  $2^{j+1}\pi$  centered about origin:

$$c_{j,m,n}^D = \sum_{k=2\pi(-2^j/2+1/2)} e^{i2^{-j}nk} \sum_{p \in 2\pi Z} \overline{\hat{\psi}_m^j(k+2^j p)} \hat{u}(k+2^j p). \quad (5)$$

A simple wrapping trick is used for the implementation of discrete wavelet packets and the steps involved are:

- Perform an FFT of size  $N$  on the samples of  $u(k)$ .
- For each pair  $(j,m)$ , wrap the product  $\overline{\hat{\psi}_m^j} \hat{u}$  by periodicity inside the interval  $[-2^j\pi, 2^j\pi]$  and perform an inverse FFT of size  $2^j$  to obtain  $c_{j,m,n}^D$ .
- Repeat step 2 for all pairs  $(j,m)$ .

The overall complexity of the algorithm is  $O(N \text{Log}N)$  and the wavelet packets are decomposed into positive and negative frequency components, represented by

$$\hat{\psi}_{m,n}^j(\omega) = \hat{\psi}_{m,n,+}^j(\omega) + \hat{\psi}_{m,n,-}^j(\omega). \quad (6)$$

Hilbert transform  $H \psi_{m,n}^j$  of (6) represents an orthonormal basis  $L^2(\mathbb{R})$  and is obtained through a linear combination of negative and positive frequency bumps weighted by  $i$  and  $-i$  respectively.

$$H \hat{\psi}_{m,n}^j(\omega) = -i \hat{\psi}_{m,n,+}^j(\omega) + i \hat{\psi}_{m,n,-}^j(\omega). \quad (7)$$

## B. 2D Discrete Wave Atoms

A two-dimensional orthonormal basis function with 4 bumps in frequency plane is formed by individually taking products of 1D wave packets. Mathematical formulation and implementations for 1D case are detailed in the previous section. 2D wave atoms are indexed by  $\mu=(j,m,n)$ , where  $m=(m_1,m_2)$  and  $n=(n_1,n_2)$ . Construction is not a simple tensor product since there is only one scale subscript  $j$ . This is similar to the non-standard or multi-resolution analysis wavelet bases where the point is to enforce same scale in both directions in order to retain an isotropic aspect ratio. Equation (1) is modified in 2D as:

$$\varphi_{\mu}^+(x_1, x_2) = \psi_{m_1}^j(x_1 - 2^{-j}n_1) \psi_{m_2}^j(x_2 - 2^{-j}n_2). \quad (8)$$

The Fourier transform of (8) is separable and its dual orthonormal basis is defined by Hilbert transformed wavelet packets in (10).

$$\hat{\varphi}_{\mu}^+(\omega_1, \omega_2) = \hat{\psi}_{m_1}^j(\omega_1) e^{-i2^{-j}n_1\omega_1} \hat{\psi}_{m_2}^j(\omega_2) e^{-i2^{-j}n_2\omega_2}. \quad (9)$$

$$\varphi_{\mu}^-(x_1, x_2) = H \psi_{m_1}^j(x_1 - 2^{-j}n_1) H \psi_{m_2}^j(x_2 - 2^{-j}n_2). \quad (10)$$

Combination of (8) and (10) provides basis functions with two bumps in the frequency plane, symmetric with respect to the origin and thus directional wave packets oscillating in a single direction are generated.

$$\varphi_{\mu}^{(1)} = \frac{\varphi_{\mu}^+ + \varphi_{\mu}^-}{2}, \quad \varphi_{\mu}^{(2)} = \frac{\varphi_{\mu}^+ - \varphi_{\mu}^-}{2}. \quad (11)$$

$\varphi_{\mu}^{(1)}$  and  $\varphi_{\mu}^{(2)}$  together form the wave atom frame and are jointly denoted by  $\varphi_{\mu}$ . Wave atom algorithm is based on the apparent generalization of the 1D wrapping strategy to two dimensions and its complexity is  $O(N^2 \text{Log}N)$ .

## VECTOR QUANTIZATION USING SOM

Self organizing map (SOM) is a type of artificial neural network that is trained using an unsupervised learning technique to produce a low-dimensional representation of the input space. SOM performs vector quantization of the significance map and improves bit rate of the reconstructed image. A SOM is used for vector quantization since it generates a mapping from a continuous input space to a discrete output space while simultaneously preserving the topological properties of input data i.e. points close to one another in the input space are mapped to same or neighboring neurons in the output layer. The SOM is a fully connected single-layer linear network as shown in Fig. 2. Neighborhood information of input space is retained at output layer through tuned weights. The output node that results in minimum Euclidean distance between its weight vector and the present input vector wins the competition. A neuron  $l$  wins if its weight vector is the closest to an input vector  $m$  as:

$$Neuron_i = \sqrt{\sum_{l=1}^n (W_{li} - x_i)^2}, \quad (12)$$

where  $W_{li}$  is the weight connecting input element  $i$  to the output neuron  $l$ . Unlike a simple competitive network, in SOM the winner of a competition and its neighbors has their weights updated according to the Kohonen [17] update rule,

$$W_i(q) = (1 - \alpha)W_i(q-1) + \alpha X(q), \quad (13)$$

$$i \in N_{i^*}(d),$$

where  $\alpha$ ,  $q$ ,  $X$  represent learning rate, time instance and input data respectively. The neighborhood  $N_{i^*}(d)$  contains indices for all neurons that lie within a radius  $d$  of the winning neuron  $i^*$ .

$$N_i(d) = \{j, d_{ij} \leq d\}. \quad (14)$$

Different steps involved in quantization of a significance map are:

- The significance map is divided into blocks of size  $m \times m$  to be used for training although the blocks which contain all zeros or ones are excluded. This adaptation is performed to minimize the training time.
- The synaptic weights and blocks are randomly chosen and a neural network with only  $n-2$  neurons is selected.
- The training of a network is carried out according to the learning rule described above and weights are updated according to eq. (13).
- After training, two neurons with weights equal to all zeros and ones are appended to compensate for blocks excluded in the training process.

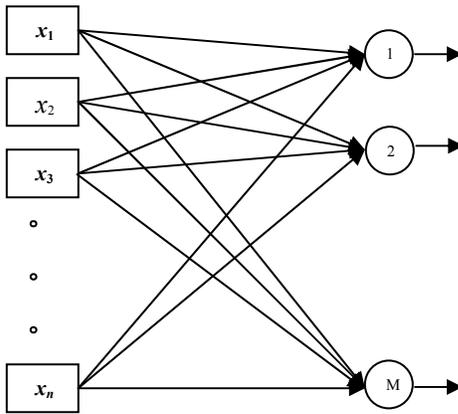


Figure 2. Architecture of Kohonen Neural Network.

## PROPOSED IMAGE COMPRESSION ALGORITHM

Wave atom decomposition is used for sparse representation of fingerprint images since they belong to a category of images that oscillate smoothly in varying directions. Various steps involved in the proposed method are shown in Table II. Discrete 2D wave atom decomposition is applied on the original fingerprint image to efficiently capture coherence patterns along and across the oscillations. Fingerprint images are digitized using 256 gray levels therefore a transformation in color space is not required. No further pre-processing is performed to preserve the fine detail in an image. An orthonormal basis  $\phi_\mu^+(x)$  ( $\phi_\mu^{(1)} + \phi_\mu^{(2)}$ ) is used instead of a tight frame since each basis function oscillates in two distinct directions instead of one. This orthobasis variant property is significantly important in applications where redundancy is undesired.

Magnitudes of wave atom decomposed coefficients, carrying low information content, are either zero or very close to zero hence they can be discarded without a substantial degradation in an image quality. An appropriate global threshold is used to achieve desired transmission bit rate. After thresholding significance map matrix and a significant coefficient vector is generated. Significance map is a matrix of binary values that indicates the presence or absence of significant coefficient at a specific location. The significance map is processed with a sequence of dilation and closing operations with a circular structuring element of radius 3. Dilations and erosions were performed to minimize the loss of potentially important coefficients due to thresholding thereby ensuring a high peak signal to noise ratio (PSNR) of the reconstructed image.

TABLE II. PROPOSED FINGERPRINT IMAGE COMPRESSION USING WAVE ATOMS AND SOM

**INPUT:** fingerprint images, bit rate or compression ratio, size and type of structuring element, block size, number of codewords, quantization levels.

**OUTPUT:** Compressed fingerprint image.

1. Apply 2D discrete wave atoms decomposition to the original fingerprint image.
2. Threshold wave atoms coefficients to achieve desired compression ratio.
3. Generate binary significance map and significant coefficients.
4. Preprocess significance map with dilations and erosions with the given structuring element.
5. Divide significance map into non-overlapping block and vector quantize with desired number of codewords.
6. Scalar quantize significant coefficients with required quantization levels.
7. Encode significance map and significant coefficients with arithmetic entropy coding.
8. Reconstruct image by decoding, un-quantizing and performing inverse 2D wave atoms decomposition and obtain mean square error and peak signal to noise ratio.

The significance map is divided into non-overlapping blocks of 4x4. These non-overlapping blocks of significance map are vectorized and quantized using SOM with 64 output neurons. Small blocks of data are used in order to minimize the error during vector quantization. The significant coefficients are quantized using a uniform scalar quantizer with 512 distinct levels.

The quantized significance map and significant coefficients are transformed using an arithmetic encoder. Arithmetic coding is a variable length entropy scheme that attempts to minimize the number of bits by converting a string into another representation using more bits for infrequent characters and *vice versa*. As opposed to other encoding techniques which convert the input message into the component symbols and replace each symbol with a code word; arithmetic coding represents the entire message into a single number thereby achieving optimal entropy encoding.

### RESULTS AND DISCUSSION

Various fingerprint images used in FBI's WSQ standard are compressed using the proposed method and substantial improvement in results is achieved. The quality of various image compression techniques depends upon how close is the reconstructed image to the original image. Different metrics have been proposed to investigate the quality of compression algorithms. Some methods investigate similarity while others explore the level of dissimilarity between reconstructed and the reference image.

Mean square error (MSE) and peak signal to noise ratio (PSNR) are two celebrated metrics used to examine the qualitative performance. MSE is a distortion metric that provides a measure of dissimilarity between two images. PSNR represents a ratio of maximum possible power of a signal and the power of the corrupting noise/distortion that affects the fidelity of its representation.

$$MSE = \frac{1}{R * C} \sum_{i=1}^R \sum_{j=1}^C |I(i, j) - I'(i, j)|^2, \quad (15)$$

$$PSNR = 10 \log_{10} \frac{255^2}{MSE}, \quad (16)$$

where  $R$  indicates the number of image rows and  $C$  refers to the number of columns,  $I(\cdot)$  represents the original image whereas  $I'(\cdot)$  refers to the reconstructed image.

Fig. 3 shows one of the sample fingerprint image used in evaluating the performance of the proposed compression algorithm. Figs. 4-5 show the compressed image at a compression ratio (CR) of 12.9 using FBI's wavelet scalar quantization and the proposed wave atoms based scheme respectively. It is evident from Fig. 5 that the proposed method using wave atoms decomposition does a better job in preserving the fine details in a fingerprint image (i.e. the minutiae - ridges ending and bifurcations) and contains less blur in comparison with FBI's WSQ standard. Table III compares the PSNR obtained using our proposed method with

the WSQ at different compression ratios. As shown in Table III fingerprint compression based on wave atoms decomposition produces a significant improvement in PSNR at high compression ratios (low bit rates).



Figure 3. Sample FBI Fingerprint Image



Figure 4. Image Compressed using FBI's WSQ Compression Standard at CR=12.9:1



Figure 5. Image Compressed using the Proposed Wave Atoms Based Compression Algorithm at CR=12.9:1

We compare the performance of our proposed method with WSQ at higher compression, since most compression standards perform plausibly well at lower compression rates but their performance drastically deteriorates at higher values. For CR 19:1, genetic algorithm based multiresolution analysis algorithm [18] achieves maximum MSE improvement of 16.71% and a PSNR gain of 0.794 db over the FBI fingerprint compression standard. However our proposed algorithm offers a maximum PSNR gain of 6.04 db and MSE improvement of 18.1% at a compression ratio of 32:1 compared to WSQ.

TABLE III. COMPRESSION RATIO VS. PSNR FOR FBI'S WSQ AND THE PROPOSED METHOD

Compression Ratio	Bit rate	FBI's WSQ [3] PSNR (db)	Proposed PSNR (db)
80	0.1	23.72	29.32
53.3	0.15	25.15	30.72
40	0.2	26.25	32.26
32	0.25	27.14	33.18
26.6	0.3	27.96	33.65

### CONCLUSION

Application of wave atoms based decomposition for fingerprint image compression results into a significant improvement in PSNR compared to FBI's WSQ fingerprint compression standard. The improvements in PSNR are more pronounced and distinct at higher compression ratios and validate the fact that wave atoms multiresolution analysis offers significantly sparser expansion, for oscillatory functions, than other fixed standard representations like wavelets, curvelets and Gabor atoms and captures coherence of pattern both across and along oscillations. Law enforcement, multimedia, and data mining related applications can benefit from our proposed compression scheme.

### ACKNOWLEDGMENT

This research has been supported in part by the Canada Research Chair Program and the NSERC discovery grant. The authors would also like to extend their gratitude to wavelet.org team for helpful data and links.

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